Week 11: Eigenvalue / Eigenvector.

Recall

Defn: Given a $n\times n$ matrix A, a vector $v \in \mathbb{R}^n$ is said to be a eigenvector of A with eigenvalue $\lambda \in \mathbb{R}$ if

$$Av = \lambda v.$$

Rmk from last week:

1) To find v and λ, we find $\lambda \in \mathbb{R}$ s.t. $A - \lambda I$ is singular.

   Then v is element in $\mathrm{Null}(A-\lambda I)$.

2) Some matrix does not have eigenvector.

3) The subset $E = \{v \in \mathbb{R}^n \mid Av = \lambda v\}$ is a subspace.

Thm: Let A be a $n\times n$ matrix, $v_1, v_2, \dots, v_k$ be eigenvectors with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k$, then $v_1, v_2, \dots, v_k$ are linearly indep.

pf:

pf by induction: If $k=1$, trivially true.

If the conclusion holds for $k_0 > 0$.

Let $\alpha_1, \alpha_2, \dots, \alpha_{k_0+1} \in \mathbb{R}$ be s.t.

$$\sum_{i=1}^{k_0+1} \alpha_i v_i = 0$$

$$\Rightarrow \sum_{i=1}^{k_0} \alpha_i \lambda_i v_i = -\alpha_{k_0+1} \lambda_{k_0+1} v_{k_0+1} = \lambda_{k_0+1}\left(\sum_{i=1}^{k_0} \alpha_i v_i\right)$$

$$\Rightarrow \sum_{i=1}^{k_0} (\alpha_i \lambda_i - \alpha_i \lambda_{k_0+1}) v_i = 0$$

$$\Rightarrow \alpha_i(\lambda_i - \lambda_{k_0+1}) = 0 \quad \forall i = 1, 2, \cdots, k_0$$

$$\Rightarrow \alpha_i = 0 \quad \forall i = 1, 2, \cdots, k_0, \underline{\underline{k_0+1}} \quad \#$$

By MI, the conclusion holds.

__Corollary__: If A be a $n \times n$ matrix then A can have at most n distinct eigenvalues.

__pf:__ If A has distinct eigenvalue $\lambda_1, \lambda_2, \cdots, \lambda_k$ with eigenvector $(v_i)_{i=1}^{k}$

then $\text{span}\{v_1, v_2, \cdots, v_k\}$ is a subspace of $\mathbb{R}^n$

$\Rightarrow k \leq n$.

---

implication

Eg: $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $w = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ ← eigenvectors

eigenvalues → 1, 2, 3

Corollary ⇒ these are the only eigenvalues

distinct ⇒ $\text{span}\{u, v, w\}$ has $\dim = 3 = \dim(\mathbb{R}^3)$

$\therefore \text{span}\{u, v, w\} = \mathbb{R}^3$.

__Basic Eg__: $A = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$ i.e. $A_{ij} = \begin{cases} \lambda_i & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$

Notation: $A = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$

Interested in those A which are similar to diagonal matrix.

Defn: Let A be a $n \times n$ matrix

(a) Suppose U is a $n \times n$ invertible matrix, $U^{-1}AU$ is said to be a diagonalization of A iff $U^{-1}AU$ is diagonal

(b) A is called diagonalizable iff $\exists\, U$, $n \times n$ invertible s.t. $U^{-1}AU$ is a diagonalization of A.

Q: How to find $\mathcal{U}$ above??

Eg: $A = \begin{bmatrix} 13 & 30 \\ -6 & -14 \end{bmatrix}$ find $\mathcal{U}$ and eigenvalues (if exists).

$A - \lambda I = \begin{bmatrix} 13-\lambda & 30 \\ -6 & -14-\lambda \end{bmatrix}$ is singular if rank $(A - \lambda I) = 1$.

(if $= 2 \Rightarrow$ non-singular)

i.e. $\begin{cases} 13-\lambda = 30\beta \\ -6 = (-14-\lambda)\beta \end{cases}$ for some $\beta \in \mathbb{R}$, $\beta \neq 0$

Solving quadratic equ $\Rightarrow \lambda = 1$ or -2.

$u = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ $\quad v = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$u \in \mathrm{Null}(A - I)$; $v \in \mathrm{Null}(A + 2I)$.

Goal: Looking for $\mathcal{U}$ s.t. $\mathcal{U}^{-1}A\,\mathcal{U} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$.

i.e. $\mathcal{U}^{-1}A\,\mathcal{U} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathcal{U}^{-1}A\,\mathcal{U} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lambda_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Leftrightarrow$ $A\left(\mathcal{U} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \lambda_1 \left(\mathcal{U} \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $A\left(\mathcal{U} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \lambda_2 \left(\mathcal{U} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

Hope: $\begin{cases} U\begin{bmatrix}1\\0\end{bmatrix} = u & \text{where } Au = u \\ U\begin{bmatrix}0\\1\end{bmatrix} = v. & \text{where } Av = -2v. \end{cases}$

Hence $U = [u, v] = \begin{bmatrix} 5 & 2 \\ -2 & 4 \end{bmatrix}$

<u>Checking</u>: • U is invertible since $\{u, v\}$ are linearly indep

• $Ue_1 = u$, $Ue_2 = v$.

<u>Eg</u>: $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, $u_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $u_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $u_3 = \begin{bmatrix}3\\4\\2\end{bmatrix}$ ← eigenvectors

$\qquad\qquad\qquad\qquad 1 \qquad\qquad 2 \qquad\qquad 3$ ← eigenvalues

write $G = [u_1 u_2 u_3] = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ is invertible.

• $AGe_i = Au_i = \lambda_i u_i = \lambda_i G e_i$

$\Rightarrow (G^{-1} A G) e_i = \lambda_i e_i \Rightarrow$ diagonal

(why: If B is $n \times n$ matrix, then the k-th column of B = $B \cdot e_k$)

Finding G^{-1}: $\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \to [I_3 \mid G^{-1}]$

$$G^{-1} = \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 0 & 1 & -2 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

<u>Checking</u>: $G^{-1} A G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

<u>True in general!!</u>

Thm : Let A be $n\times n$ matrix, $u_1, u_2, \cdots, u_n$ be vector on $\mathbb{R}^n$ and $G = [u_1\ u_2 \cdots u_n]$ is non-singular. Then the following are equivalent

(or they form a basis of $\mathbb{R}^n$)

① $u_1, u_2, \cdots, u_n$ are eigenvector of A with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ respectively.

② $G^{-1} A G = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

Corollary: If A = $n\times n$ matrix has n distinct eigenvalues, then A is diagonalizable.

pf: By assumption, $\exists$ eigenvectors $u_1, u_2, \cdots, u_n$ which are linearly indep $\Rightarrow$ $\operatorname{span}\{u_1, u_2, \cdots, u_n\} = \mathbb{R}^n$.

$\Rightarrow$ G exists with $G^{-1} A G =$ diagonal.
 (invertible)

---

observation: Suppose A = $n\times n$ matrix, then A is singular iff 0 = eigenvalue of A.

(And $\vec{v} \neq 0$, $\vec{v} \in \operatorname{Null}(A) \Rightarrow v$ = eigenvector of A with eigenvalue $= 0$)

Theorem: Let A be $n\times n$ matrix. Suppose $\lambda_1, \lambda_2, \cdots, \lambda_k$ are all eigenvalues of A. Then the following are equivalent.

① A is diagonalizable (i.e. $\exists\ U$ invertible s.t. $U^{-1} A U =$ diagonal)

② $\sum_{i=1}^{k} \dim(E_A(\lambda_i)) = n$.

Moreover, if ② holds, take $\{v_{ij}\}_{j=1}^{\dim(E_A(\lambda_i))}$ to be basis for $E_A(\lambda_i)$, then

$\{v_{ij} \mid i=1,2,\dots,k;\ j=1,2,\dots,\dim(E_A(\lambda_i))\}$ forms a basis of $\mathbb{R}^n$.

Eg: 1) $A=\begin{bmatrix}13 & 30\\ -6 & -14\end{bmatrix}$; $u_1=\overset{1}{\begin{bmatrix}5\\-2\end{bmatrix}}$; $u_2=\overset{-2}{\begin{bmatrix}2\\-1\end{bmatrix}}$

$E_A(1)=\text{span}\{u_1\}$ $\quad E_A(-2)=\text{span}(u_2)$

$\dim(E_A(1))+\dim(E_A(-2)) = 2 = \dim(\mathbb{R}^2)$.

$\Rightarrow$ diagonalizable $\left(\text{in fact } \begin{bmatrix}5 & 2\\ -2 & -1\end{bmatrix}^{-1} A \begin{bmatrix}5 & 2\\ -2 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -2\end{bmatrix}\right)$

---

2) $A=\begin{bmatrix}1 & 1 & 1\\ 0 & 2 & 2\\ 0 & 0 & 3\end{bmatrix}$. $u_1=\overset{1}{\begin{bmatrix}1\\0\\0\end{bmatrix}}$, $u_2=\overset{2}{\begin{bmatrix}1\\1\\0\end{bmatrix}}$, $u_3=\overset{3}{\begin{bmatrix}3\\4\\2\end{bmatrix}}$.

linearly independent.

$\dim(E_A(1))+\dim(E_A(2))+\dim(E_A(3)) = 3 = \dim(\mathbb{R}^3)$.

$\Rightarrow$ A is diagonalizable.

---

3) $A=\begin{bmatrix}2 & 1 & 0\\ 0 & 2 & 1\\ 0 & 0 & 2\end{bmatrix}$. $A-\lambda I$ is singular iff $\lambda=2$

$\Rightarrow$ eigenvalue $= 2$ only.

$E_A(2) = \text{span}\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\quad\therefore \dim(E_A(2)) = 1 < \dim(\mathbb{R}^3) = 3$.

∴ Not diagonalizable.

---

4) $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ has no eigenvalues $\Rightarrow$ Not diagonalizable.

☆ **prp**: Let A be $n \times n$ matrix which is diagonalizable with

$U^{-1}AU$ = diagonal matrix D for some invertible U.

Then ① $\forall m \in \mathbb{N}$, A^m is diagonalizable with

$$U^{-1}A^mU = D^m = \text{diagonal}.$$

② If A = invertible, then A^{-1} is diagonalizable.

---

pf: ① $(U^{-1}AU)^m = (U^{-1}AU)(U^{-1}AU)\cdots(U^{-1}AU)$

$$= U^{-1}A^mU$$

$(U^{-1}AU)^m = D^m$ (= diagonal) ※

② $U^{-1}AV = D = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$

∵ A = invertible ∴ diagonal entry of $D \neq 0$.

∴ D is invertible with $D^{-1} = \operatorname{diag}(\lambda_1^{-1}, \cdots, \lambda_n^{-1})$

$$D^{-1} = (U^{-1}AU)^{-1} = U^{-1}A^{-1}U \ \#.$$

Question: How to determine singularity of a matrix ??

☆☆ Finding RREF of A is too much !! (for this purpose)

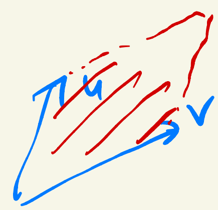
Motivation: Find a measurement of a matrix.

$$\boxed{\text{Size of } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = [u \ v \ w] \ ??}$$

- When size $= 0 \Rightarrow$ singular
- size(AB) = size$(A) \cdot$ size(B)

From vector point of view:

2D: $A = [u, v]$



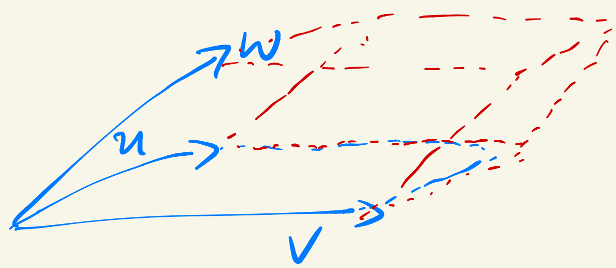
"size" of A ? Area of 

$= |u_1 v_2 - u_2 v_1|$ (using coordinate geom)

Expectation on orientation: size$\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 1 = -$size$\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$ ....

3D:

$A = [u \ v \ w]$:



"size" of A (Best guess)

using coordinate geom.

$= $ volume of parallelogram $= \left| \begin{aligned} &u_1(v_2 w_3 - v_3 w_2) \\ &- u_2(v_1 w_3 - v_3 w_1) \\ &+ u_3(v_1 w_2 - v_2 w_1) \end{aligned} \right|$

higher dimen??

??

Ans to these = determinant of A, $\underline{\underline{\det(A)}}$.

Defn: Given a $n \times n$ matrix A and $k, l \in \{1, 2, \cdots, n\}$.

$A(k|l) = (n-1)\times(n-1)$ matrix obtained by removing the k-th row, l-th column of A.

Eg: $A = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$, $A(1|1) = \begin{bmatrix} v_2 & w_2 \\ v_3 & w_3 \end{bmatrix}$

$A(2|3) = \begin{bmatrix} u_1 & v_1 \\ u_3 & v_3 \end{bmatrix}$, etc.

Defn: Given a $n\times n$ matrix A with entry $= A_{ij}$

(a) if $n=1$, $\det A = A_{11}$

(b) if $n>1$, define $\det A$ inductively on n by.

$$\det(A) = \sum_{j=1}^{n} (-1)^{j+1} A_{j1} \cdot \det(A(j|1)).$$

Eg: 1) $A = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}$, $\det(A) = u_1 \cdot v_2 - u_2 v_1$

2) $A = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$, $\det(A) = u_1 \cdot \det\begin{bmatrix} v_2 & w_2 \\ v_3 & w_3 \end{bmatrix} = u_1(v_2 w_3 - v_3 w_2)$

$- u_2 \det\begin{bmatrix} v_1 & w_1 \\ v_3 & w_3 \end{bmatrix}$ $\quad -u_2(v_1 w_3 - v_3 w_1)$

$+ u_3 \det\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$ $\quad +u_3(v_1 w_2 - v_2 w_1)$

#

Keep in Mind: There are lots of symmetry in computing $\det(A)$.

Lemma: For $n>1$, and $k=1,2,\cdots,$ or n.

$$\det A = \sum_{j=1}^{n} (-1)^{j+k} A_{jk} \det(A(j|k)).$$

Eg: $A = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$, then $\det A = -v_1(u_2w_3 - u_3w_2) + v_2(u_1w_3 - u_3w_1) - v_3(u_1w_2 - u_2w_1)$.

Lemma: We also have $\det(A^t) = \det(A)$.

i.e. $\det A = \sum_{j=1}^{n} (-1)^{j+k} A_{kj} \det(A(k|j)) = \det(A^t)$

Ok. But why care ?? (Before we study them further)

$\boxed{\text{Thm: Given a } n\times n \text{ matrix } A, \text{ then } A = \text{non-singular iff } \det(A) \neq 0.}$

Therefore, to find the eigenvalue of A, we only need to solve $\lambda \in \mathbb{R}$ s.t. $\underbrace{\det(A-\lambda I)}_{\text{polynomials}} = 0$

Verify it Now: relies on symmetry

Thm (linearity in columns) Suppose A is a $n\times n$ matrix s.t.

$A = [a_1 \ a_2 \ \cdots \ a_k \cdots \ a_n]$

$B = [b_1 \ b_2 \ \cdots \ b_k \cdots \ b_n]$ where $a_i, b_i, c_i \in \mathbb{R}^n$.

$C = [c_1 \ c_2 \ \cdots \ c_k \cdots \ c_n]$

If $\cdot\ a_i = b_i = c_i \quad \forall i \neq k$

$\cdot\ a_k = \lambda b_k + c_k$

then $\det A = \lambda \det B + \det C$.

i.e.

$$\det \begin{bmatrix} a_1 & a_2 & \cdots & \boxed{\lambda b_k + c_k} & \cdots & a_n \end{bmatrix}$$

$$= \det [a_1 \cdots b_k \cdots a_n] \cdot \lambda$$

$$+ \det [a_1 \cdots c_k \cdots a_n].$$

Since $\det(A^t) = \det(A)$, linearity also holds along rows:

Thm: Suppose $A = n \times n$ matrix, then

$$\det \begin{bmatrix} r_1 \\ \vdots \\ r_{k-1} \\ \lambda s_k + t_k \\ r_{k+1} \\ \vdots \\ r_n \end{bmatrix} = \det \begin{bmatrix} r_1 \\ \vdots \\ r_{k-1} \\ s_k \\ r_{k+1} \\ \vdots \\ r_n \end{bmatrix} \cdot \lambda + \det \begin{bmatrix} r_1 \\ \vdots \\ r_{k-1} \\ t_k \\ r_{k+1} \\ \vdots \\ r_n \end{bmatrix}$$

where $r_i^t \in \mathbb{R}^n$ and $s_k^t, t_k^t \in \mathbb{R}^n$.

Rmk: Putting $\lambda = -1$, we have

Thm: $\det [a_1 \cdots a_{k-1}\ -a_k\ a_{k+1} \cdots a_n] = -\det [a_1 \cdots a_k \cdots a_n]$

eg: $\det \begin{bmatrix} u_1 & -v_1 & w_1 \\ u_2 & -v_2 & w_2 \\ u_3 & -v_3 & w_3 \end{bmatrix} = -\det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$

Thm: $\det [a_1 \cdots a_{p-1}\ a_q\ a_{p+1} \cdots a_{q-1}\ a_p\ a_{q+1} \cdots a_n]$

$= -\det [a_1 \cdots a_{p-1}\ a_p\ a_{p+1} \cdots a_{q-1}\ a_q\ a_{q+1} \cdots a_n]$

(anti-symmetric upon interchanging column)

eg: $\det \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} = -\det \begin{bmatrix} w_1 & v_1 & u_1 \\ w_2 & v_2 & u_2 \\ w_3 & v_3 & u_3 \end{bmatrix}$

||

$\Downarrow$

Trivial case of Thm: Given a $n\times n$ matrix A.

① If there are two identical columns of A, then $\det(A)=0$.

($\Downarrow$ A is singular $\Uparrow$)

② If there is a column of A s.t. it is linear combination of others, then $\det A = 0$.

pf: case ①:

$$\det \begin{array}{c} \\ \end{array}[a_1 \cdots \overset{p\text{-th}}{a_p} \cdots \overset{q\text{-th}}{a_q} \cdots a_n]$$

$$= -\det[a_1 \cdots a_q \cdots a_p \cdots a_n]$$

$$= -\det[a_1 \cdots a_p \cdots a_q \cdots a_n] \quad \because a_p = a_q$$

$$\Rightarrow \det A = 0.$$

---

case ②: $\det A = \det[a_1 \cdots a_p \cdots a_n]$

$$\left(\text{where } a_p = \sum_{i\neq p} \lambda_i a_i \right)$$

p-th column

$$= \sum_{i\neq p} \lambda_i \det[a_1 \cdots a_i \cdots a_n]$$

$$= \sum_{i\neq p} \lambda_i \cdot 0 \quad \text{by ①}$$

$$= 0 \quad \#$$
